
LAPLACE EIGENFUNCTIONS AND DAMPED WAVE EQUATION II: PRODUCT MANIFOLDS.

by

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Abstract. — The purpose of this article is to study possible concentrations of eigenfunctions of Laplace operators (or more generally quasi-modes) on product manifolds. We show that the approach of the first author and Zworski [10, 11] applies (modulo rescaling) and deduce new stabilization results for weakly damped wave equations which extend to product manifolds previous results by Leautaud-Lerner [12] obtained for products of tori.

Résumé. — Dans cet article, on étudie les concentrations possibles des fonctions propres du Laplacien (ou plus généralement de quasi-modes) sur des variétés produit. On démontre que l'approche du premier auteur avec M. Zworski [10, 11] s'applique (modulo un changement d'échelle) et on en déduit de nouveaux résultats de stabilisation pour l'équation des ondes faiblement amortie qui généralisent au cas des variétés produits des résultats antérieurs de Leautaud-Lerner [12] obtenus dans le cas de produits de tores.

1. Notations and main results

In this work we continue our investigation [9] of concentration properties of eigenfunctions (or more generally quasimodes) of the Laplace-Beltrami operator on submanifolds and we study here the very particular setting of product manifolds.

Let $(M_j, g_j), j = 1, 2$ be two compact manifolds. We denote by $(M = M_1 \times M_2, g = g_1 \otimes g_2)$ the product, and by d_j (resp. d) the geodesic distance in M_j (resp. M). Let $q_0 \in M_2$ and

$$\Sigma = M_1 \times \{q_0\}.$$

For $\beta > 0$ we introduce

$$(1.1) \quad N_\beta = \{m = (p, q) \in M : d(m, \Sigma) < \beta\} = M_1 \times \{q \in M_2 : d_2(q, q_0) < \beta\}.$$

Our first result is the following.

Theorem 1.1. — *For any $\delta > 0$, there exists $C > 0, h_0 > 0$ such that for every $0 < h \leq h_0$ and every solution $\psi \in H^2(M)$ of the equation*

$$(h^2 \Delta_g + 1)\psi = F$$

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we have the estimate

$$(1.2) \quad \|\psi\|_{L^2(N_{h\delta})} \leq C(\|\psi\|_{L^2(N_{2h\delta} \setminus N_{h\delta})} + h^{2\delta-2}\|F\|_{L^2(N_{2h\delta})}).$$

As an application of Theorem 1.1, we consider weakly damped wave equations on a compact Riemannian manifold (\mathcal{M}, g) ,

$$(1.3) \quad (\partial_t^2 - \Delta_g + b(m)\partial_t)u = 0, \quad (u, \partial_t u)|_{t=0} = (u_0, u_1) \in H^{1+k}(\mathcal{M}) \times H^k(\mathcal{M}),$$

where $0 \leq b \in L^\infty(\mathcal{M})$, for which the energy

$$E(u)(t) = \int_{\mathcal{M}} (g_p(\nabla_g u(t, m), \nabla_g u(t, m)) + |\partial_t u(t, m)|^2) dv_g(m)$$

is decaying since $\frac{d}{dt}E(u)(t) = - \int_{\mathcal{M}} b(m)|\partial_t u(t, m)|^2 dv_g(m) \leq 0$. Let

$$\omega = \cup \{U \text{ open} : \text{ess inf}_U b > 0\}$$

be the domain where effective damping occurs. We denote by

$$\mathcal{GC} = \{\rho \in S^*\mathcal{M} : \exists s \in \mathbb{R}; \Phi(s)\rho = (m_1, \xi_1) \in S^*\omega\},$$

the (open) set of geometrically controlled points (here $\Phi(s)$ is the bicharacteristic flow). Let

$$(1.4) \quad \mathcal{T} = S^*\mathcal{M} \setminus \mathcal{GC}, \quad T = \Pi_x \mathcal{T}$$

where \mathcal{T} is the trapped set and Π_x the projection on the base manifold \mathcal{M} .

Our second result is the following.

Theorem 1.2. — Assume that

1. there exists a neighborhood V of T in \mathcal{M} , a compact Lipschitz Riemannian manifold (M_1, g_1) of dimension k and a Lipschitz isometry

$$\Theta : V \rightarrow (M_1 \times B(0, 1), \tilde{g} = g_1 \otimes g_2)$$

where $B(0, 1)$ is the unit ball in \mathbb{R}^{d-k} endowed with the (Lipschitz) metric g_2 ,

2. there exists $\gamma > 0, c, C > 0$ such that

$$(1.5) \quad c|z|^{2\gamma} \leq b(\Theta^{-1}(p, z)) \leq C|z|^{2\gamma}, \quad \forall (p, z) \in M_1 \times B(0, 1).$$

Then there exists $C > 0$ such that for any $(u_0, u_1) \in H^2(\mathcal{M}) \times H^1(\mathcal{M})$, the solution u to (1.3) satisfies

$$E(u)^{1/2}(t) \leq \frac{C}{t^{1+\frac{1}{\gamma}}} (\|u_0\|_{H^2(\mathcal{M})} + \|u_1\|_{H^1(\mathcal{M})}).$$

Remark 1.3. — A simpler (but weaker) statement would be to assume

- (i) $(\mathcal{M}, g) = (M_1 \times M_2, g_1 \otimes g_2)$, $q_0 \in M_2$, $T = \Sigma = M_1 \times \{q_0\}$,
- (ii) $cd(m, \Sigma)^{2\gamma} \leq b(m) \leq Cd(m, \Sigma)^{2\gamma} \quad \forall m \in M_1 \times U$.

It is classical that for non trivial dampings $b \geq 0$, the energy of solution to (1.3) converge to 0 as t tend to infinity. The rate of decay is *uniform* (and hence exponential) in energy space if and only if the *geometric control condition* [2, 7] is satisfied. In [9], we explored the question when some trajectories are trapped and exhibited decay rates (assuming more regularity on the initial data). This latter question was previously studied in a general setting in [13] and on tori in [8, 14, 1] (see also [10, 11]) and more recently by Leautaud-Lerner [12]. The geometric assumptions in [9] are much more general than in [12] which is essentially restricted to the case of product of flat tori. On the other hand, due to

this more favorable geometry, the decay rate in [12] is better than in [9]. Theorem 1.2 shows that Leautaud-Lerner's result (the better decay rate) extends straightforwardly to the case of product manifolds $(M_1 \times M_2, g = g_1 \otimes g_2)$.

- Remark 1.4.** — 1. According to Theorem 1.6 in [12] the rate of decay in t obtained in Theorem 1.2 above is optimal in general.
2. Theorem 1.1 is a propagation result in the z -variable in $B(0, 1)$, and since z is actually very close to 0, the relevant object is $g_2(0)$ (constant coefficients) rather than $g_2(z)$. The smaller δ , the further we need to propagate in the z variable and hence we better the quasi modes we need to consider (due to the worse error factor $h^{2\delta-2}$).
3. The case $\delta = 1/2$ in Theorem 1.1 is a particular case of our results in [9] (which are actually much more general and hold without the "product" assumption on the geometry). On the other hand, the results in [9] are *local*, while for $\delta < 1/2$, the estimate (1.2) is *non local*. Indeed, trying to replace ψ by $\chi\psi$ will add to the r.h.s. a term $([h^2\Delta, \chi]\psi)$ which is clearly bounded in L^2 by $O(h)$, giving an error of order $O(h^{2\delta-1}) \gg 1$ to the final result. On the other hand, as soon as $\delta < 1/2$, estimate (1.2) is false without the product structure assumption as can be easily seen on spheres by considering the eigenfunctions $e_n = (x_1 + ix_2)^n$ with eigenvalues $\lambda_n = n(n + d - 1) = h_n^{-2}$ which concentrate in an $h_n^{1/2}$ -neighborhood of the equator

$$E = \{x \in \mathbb{R}^{d+1} : |x| = 1, x_3 = \dots = x_{d+1} = 0\}.$$

In this case, we get $(h_n^2\Delta + 1)e_n = 0$, but

$$\|e_n\|_{L^2(N_{h_n^\delta})} \sim C_1 h_n^{\frac{d-1}{4}}, \quad \|e_n\|_{L^2(N_{2h_n^\delta} \setminus N_{h_n^\delta})} \leq C_2 e^{-ch_n^{2\delta-1}}, \quad n \rightarrow +\infty,$$

contradicting (1.2) since $2\delta - 1 < 0$.

4. No smoothness is assumed on the function $b \in L^\infty(\mathcal{M})$. Notice however that (contrarily to the results in [9]) the lower bound in (1.5) is *not sufficient* (at least with our approach) and we do need also the upper bound.
5. As will appear clearly in the proof, we could assume that T is isometric to finitely many product manifolds, with possibly different constants γ , the final decay rate being given by the largest γ .

The paper is organized as follows. We first show how to deduce from Theorem 1.1 a resolvent estimate which according to previous works by Borichev-Tomilov imply Theorem 1.2. Then we prove Theorem 1.1 by elementary scaling and propagation arguments.

2. From concentration to stabilization results (Proof of Theorem 1.2)

According to the works by Borichev-Tomilov [3], stabilization results for the wave equation are equivalent to resolvent estimates. As a consequence, to prove Theorem 1.2, it is enough to prove (see [12, Proposition 1.5])

Proposition 2.1. — *We keep the geometric assumptions in Theorem 1.2. Consider for $h > 0$ the operator*

$$(2.1) \quad L_h = -h^2\Delta_g - 1 + ihb, \quad b \in L^\infty(\mathcal{M}).$$

Then there exist $C > 0, h_0 > 0$ such that for all $0 < h \leq h_0$

$$\|\varphi\|_{L^2(\mathcal{M})} \leq Ch^{-1-\frac{\gamma}{\gamma+1}} \|L_h\varphi\|_{L^2(\mathcal{M})},$$

for all $\varphi \in H^2(\mathcal{M})$.

Proof. — We start with a simple *a priori estimate*. Multiplying both sides of the equation

$$(2.2) \quad (-h^2\Delta_g - 1 + i\hbar b)\varphi = f.$$

by $\overline{\varphi}$, integrating by parts on \mathcal{M} and taking real and imaginary parts gives

$$(2.3) \quad \hbar \int_{\mathcal{M}} b(m) |\varphi(m)|^2 dv_g(m) \leq \|\varphi\|_{L^2(\mathcal{M})} \|f\|_{L^2(\mathcal{M})},$$

$$(2.4) \quad \hbar^2 \int_{\mathcal{M}} g_m(\nabla_g \varphi(m), \overline{\nabla_g \varphi(m)}) dv_g(m) \leq \|\varphi\|_{L^2(\mathcal{M})}^2 + \|\varphi\|_{L^2(\mathcal{M})} \|f\|_{L^2(\mathcal{M})}.$$

Now, in the neighborhood V of T we use our isometry Θ and we set

$$(2.5) \quad u(p, z) = \varphi(\Theta^{-1}(p, z)), \quad \tilde{b}(p, z) = b(\Theta^{-1}(p, z)), \quad \tilde{f}(p, z) = f(\Theta^{-1}(p, z)).$$

Then from (2.2) we obtain the equation on $M_1 \times B(0, 1)$

$$(h^2\Delta_{\tilde{g}} + 1)u = i\hbar\tilde{b}u - \tilde{f}.$$

We can therefore apply Theorem 1.1 and we obtain

$$(2.6) \quad \|u\|_{L^2(M_1 \times \{|z| \leq h^\delta\})} \leq C\|u\|_{L^2(M_1 \times \{h^\delta \leq |z| \leq 2h^\delta\})} + Ch^{2\delta-2}\|i\hbar\tilde{b}u - \tilde{f}\|_{L^2(M_1 \times \{|z| \leq 2h^\delta\})}.$$

On the other hand, from (2.3), and the lower bound in assumption (1.5), we deduce

$$(2.7) \quad \|u\|_{L^2(M_1 \times \{h^\delta \leq |z|\})}^2 \leq Ch^{-1-2\delta\gamma}\|\varphi\|_{L^2(\mathcal{M})}\|f\|_{L^2(\mathcal{M})}$$

while from the upperbound in assumption (1.5), we get

$$(2.8) \quad \begin{aligned} \|i\hbar\tilde{b}u\|_{L^2(M_1 \times \{|z| \leq 2h^\delta\})}^2 &\leq h^2 \left(\sup_{M_1 \times \{|z| \leq 2h^\delta\}} |\tilde{b}| \right) \|\tilde{b}^{1/2}u\|_{L^2(M_1 \times \{|z| \leq 2h^\delta\})}^2 \\ &\leq Ch^{1+2\delta\gamma}\|\varphi\|_{L^2(\mathcal{M})}\|f\|_{L^2(\mathcal{M})}. \end{aligned}$$

Gathering (2.6), (2.7) and (2.8) we obtain,

$$(2.9) \quad \begin{aligned} \|u\|_{L^2(M_1 \times B(0,1))}^2 &\leq Ch^{-1-2\delta\gamma}\|\varphi\|_{L^2(\mathcal{M})}\|f\|_{L^2(\mathcal{M})} \\ &\quad + Ch^{4\delta-4} \left(h^{1+2\delta\gamma}\|\varphi\|_{L^2(\mathcal{M})}\|f\|_{L^2(\mathcal{M})} + \|f\|_{L^2(\mathcal{M})}^2 \right). \end{aligned}$$

Optimizing with respect to δ leads to the choice $2\delta = \frac{1}{1+\gamma}$, which gives

$$\|u\|_{L^2(M_1 \times B(0,1))}^2 \leq Ch^{-1-\frac{\gamma}{1+\gamma}}\|\varphi\|_{L^2(\mathcal{M})}\|f\|_{L^2(\mathcal{M})} + Ch^{-2-\frac{2\gamma}{1+\gamma}}\|f\|_{L^2(\mathcal{M})}^2.$$

According to (2.5) this implies

$$(2.10) \quad \|\varphi\|_{L^2(V)} \leq Ch^{-1-\frac{\gamma}{1+\gamma}}\|\varphi\|_{L^2(\mathcal{M})}\|f\|_{L^2(\mathcal{M})} + Ch^{-2-\frac{2\gamma}{1+\gamma}}\|f\|_{L^2(\mathcal{M})}^2.$$

We can now conclude the proof of Proposition 2.1 by contradiction. If (2.1) were not true, then there would exist sequences $\varphi_n \in H^2(\mathcal{M})$, $f_n \in L^2(\mathcal{M})$, $0 < h_n \rightarrow 0$ such that

$$(-h_n^2\Delta_g - 1 + i\hbar_n b)\varphi_n = f_n, \quad \|\varphi_n\|_{L^2(\mathcal{M})} > \frac{n}{h_n^{1+\frac{\gamma}{1+\gamma}}}\|f_n\|_{L^2(\mathcal{M})}.$$

Dividing φ_n by its L^2 -norm, we deduce

$$(2.11) \quad \|\varphi_n\|_{L^2(\mathcal{M})} = 1, \quad \|f_n\|_{L^2(\mathcal{M})} = o(h_n^{1+\frac{\gamma}{1+\gamma}}), \quad n \rightarrow +\infty,$$

and from (2.10) we get

$$(2.12) \quad \lim_{n \rightarrow +\infty} \|\varphi_n\|_{L^2(V)} = 0.$$

On the other hand, the sequence (φ_n) is bounded in $L^2(\mathcal{M})$, and extracting a subsequence, we can assume that it has a semi-classical measure μ (see e.g. [5, Théorème 2]). We recall that it means that for any symbol $a \in C_0^\infty(S^*\mathcal{M})$,

$$\lim_{n \rightarrow +\infty} (a(x, h_n D_x) \varphi_n, \varphi_n)_{L^2(\mathcal{M})} = \langle \mu, a \rangle.$$

Here, since we work locally, we quantize the symbols $a \in C_0^\infty(T^*\mathbb{R}^d)$ by taking first $\phi \in C_0^\infty(\mathbb{R}^d)$ equal to 1 near the x -projection of the support of a and

$$a(x, h D_x) u = \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}(x-y) \cdot \xi} a(x, \xi) \phi(y) u(y) dy d\xi.$$

It is classical that modulo $O(h^\infty)$ smoothing operators, the operator $a(x, h D_x)$ does not depend on the choice of ϕ .

From (2.4), the sequence (φ_n) is h_n oscillating and hence any such semi-classical defect measure has total mass $1 = \lim_{n \rightarrow +\infty} \|\varphi_n\|_{L^2(\mathcal{M})}$ (see [5, Proposition 4]). From (2.3) and (2.11) we also have (notice that $|b| \leq C|b|^{1/2}$)

$$(-h^2 \Delta - 1) \varphi_n = -i h_n b \varphi_n + f_n = o(h_n)_{L^2},$$

and consequently (see [6, Proposition 4.4]) the measure μ is invariant by the bicharacteristic flow. Since from (2.3) it is 0 on $S^*\omega$, we deduce by propagation that it is also 0 on \mathcal{GC} , and hence from (2.12) it is identically null, since $S^*(\mathcal{M}) = \mathcal{T} \cup \mathcal{GC}$. This gives the contradiction. \square

3. Concentration properties (Proof of Theorem 1.1)

Recall that we have $(M, g) = (M_1 \times M_2, g_1 \otimes g_2)$. The proof of Theorem 1.1 follows, after taking scalar products with Laplace eigenfunctions in M_1 , from a rescaling argument and standard (non trapping) resolvent estimates in M_2 . When the metric g_2 is flat, the scaling argument is straightforward, while it requires a little care in the general case (see Lemma 3.5).

Let $B(q_0, r) \subset M_2$ be the ball (for the metric d_2) of radius $r > 0$ centered at q_0 .

Proposition 3.1. — *For any $\delta > 0$, there exists $C > 0, h_0 > 0$ such that for every $0 < h \leq h_0$, every $\tau \in \mathbb{R}$, every solution $U \in H^2(M_2), G \in L^2(M_2)$ of the equation on M_2*

$$(-\Delta_{g_2} - \tau)U = G$$

we have the estimate

$$(3.1) \quad \|U\|_{L^2(B(q_0, h^\delta))} \leq C(\|U\|_{L^2(B(q_0, 2h^\delta) \setminus B(q_0, h^\delta))} + h^{2\delta} \|G\|_{L^2(B(q_0, 2h^\delta))}).$$

3.1. Proof of Theorem 1.1 assuming Proposition 3.1. — Let (e_n) be a sequence of eigenfunctions of the Laplace operator on M_1 with eigenvalues $-\lambda_n^2$ forming an $L^2(M_1)$ orthonormal basis. For $\psi \in L^2(M)$, we set $\hat{\psi}_n(q) = (\psi(\cdot, q), e_n)_{L^2(M_1)}$. Then we have $\psi(p, q) = \sum_{n \in \mathbb{N}} \hat{\psi}_n(q) e_n(p)$ and it is easy to see that with the notations in (1.1), for $r > 0$

$$(3.2) \quad \|\psi\|_{L^2(N_r)}^2 = \|\psi\|_{L^2(M_1 \times B(q_0, r))}^2 = \sum_{n \in \mathbb{N}} \|\hat{\psi}_n\|_{L^2(B(q_0, r))}^2.$$

Now taking the scalar product of the equation (1.2) with e_n we see easily that $(-h^2\Delta_{g_2} + h^2\lambda_n^2 - 1)\hat{\psi}_n = \hat{F}_n$ which can be rewritten as

$$(-\Delta_{g_2} - \tau)\hat{\psi}_n = h^{-2}\hat{F}_n, \quad \tau = h^{-2} - \lambda_n^2.$$

Applying Proposition 3.1 to this equation yields

$$\|\hat{\psi}_n\|_{L^2(B(q_0, h^\delta))}^2 \leq C(\|\hat{\psi}_n\|_{L^2(B(q_0, 2h^\delta) \setminus B(q_0, h^\delta))}^2 + h^{4\delta}\|h^{-2}\hat{F}_n\|_{L^2(B(q_0, 2h^\delta))}^2).$$

Taking the sum in n and using (3.2) we obtain the estimate (1.2).

3.2. Proof of Proposition 3.1. — Since the problem is local near q_0 , after diffeomorphism we can work in a neighborhood of the origin in \mathbb{R}_z^k and we may assume that the new metric g satisfies $g|_{z=0} = \text{Id}$. Then we make the change of variables $z \mapsto x = \frac{z}{h^\delta}$ and we set $u(x) = U(h^\delta x)$, $F(x) = G(h^\delta x)$. We obtain the equation on u

$$(-\Delta_{g^h} - h^{2\delta}\tau)u = h^{2\delta}F,$$

where g^h is the metric obtained by dilatation $g^h(x) = g(h^\delta x)$. The family (g^h) converges in C^∞ topology to the flat metric $g_0 = \text{Id}$. Proposition 3.1 will follow easily from

Proposition 3.2. — Consider a family (g_n) of metrics on $B(0, 2) \subset \mathbb{R}^k$, which converges in Lipschitz topology to the flat metric when $n \rightarrow +\infty$. Then there exists $C > 0, N_0 > 0$ such that for every $n \geq N_0$, $\tau \in \mathbb{R}$, $u \in H^2(B(0, 2))$, $f \in L^2(B(0, 2))$ solutions of the equation on $B(0, 2)$

$$(-\Delta_{g_n} - \tau)u = f$$

we have the estimate

$$(3.3) \quad \|u\|_{L^2(B(0, 1))} \leq C(\|u\|_{L^2(B(0, 2) \setminus B(0, 1))} + \frac{1}{1 + |\tau|^{1/2}}\|f\|_{L^2(B(0, 2))})$$

(notice that since g_n converges to the flat metric the choice of the metric to define the L^2 -norms above is of no importance).

Remark 3.3. — Proposition 3.2 is standard for the fixed metric $g_0 = \text{Id}$ (see e.g. [6, Section 3]), as the annulus $\{x : 1 < |x| < 2\}$ controls geometrically the ball $B(0, 1)$. As a consequence, in the special case of [12] when $g = g_0$ (and hence g_n is also the standard flat metric), the proof of Theorem 1.1 is completed. In the general case, we only have to verify that the usual proof can handle the varying metric through a perturbation argument, which is precisely what we do below. It is worth noticing that the proof below implies that the propagation estimates involved in exact controllability results which are known to hold for C^2 metrics, see [4], are actually *stable* by *small Lipschitz* perturbations of the metric.

For $r > 0$ we shall set $B_r = B(0, r) \subset \mathbb{R}^k$.

To prove Proposition 3.2 we argue by contradiction. Otherwise, there would exist sequences, $\sigma_n \rightarrow +\infty$, $(\tau_n) \subset \mathbb{R}$, $(u_n) \subset H^2(B_2)$, $(f_n) \subset L^2(B_2)$ such that

$$(3.4) \quad (-\Delta_{g_{\sigma_n}} - \tau_n)u_n = f_n,$$

$$(3.5) \quad 1 = \|u_n\|_{L^2(B_1)} > n(\|u_n\|_{L^2(B_2 \setminus B_1)} + \frac{1}{1 + |\tau_n|^{1/2}}\|f_n\|_{L^2(B_2)})$$

We now distinguish three cases

- $\liminf_{n \rightarrow +\infty} \tau_n = -\infty$ (elliptic case)
- $(\tau_n)_{N \in \mathbb{N}}$ bounded (low frequency case)

– $\limsup_{n \rightarrow +\infty} \tau_n = +\infty$ (hyperbolic case)

In the first case, working with a subsequence we may assume that $\lim_{n \rightarrow +\infty} \tau_n = -\infty$. Let $\zeta \in C_0^\infty(B_2)$ equal to 1 on $B_{3/2}$. Multiplying (3.4) by $\zeta \bar{u}_n$, integrating by parts and taking the real part gives

$$(3.6) \quad \left| \int (g_{\sigma_n}(\nabla_{g_{\sigma_n}} u_n, \overline{\nabla_{g_{\sigma_n}}(\zeta u_n)})) - \zeta \tau_n |u_n|^2 dv_{g_{\sigma_n}} \right| \leq \|u_n\|_{L^2(B_2)} \|f_n\|_{L^2(B_2)}$$

which implies (after another integration by parts)

$$(3.7) \quad \left| \int \zeta g_{\sigma_n}(\nabla_{g_{\sigma_n}} u_n, \overline{\nabla_{g_{\sigma_n}} u_n}) - \left(\tau_n \zeta + \frac{\Delta_{g_{\sigma_n}}(\zeta)}{2} \right) |u_n|^2 dv_{g_{\sigma_n}} \right| \leq \|u_n\|_{L^2(B_2)} \|f_n\|_{L^2(B_2)} = o(|\tau_n|^{1/2}), \quad n \rightarrow +\infty.$$

Since $\Delta_{g_{\sigma_n}} \zeta$ is supported in $\{1 \leq |x| \leq 2\}$ and $\|u_n\|_{L^2(1 < |x| < 2)} = o(1)$, we deduce if $\tau_n \rightarrow -\infty$

$$\lim_{n \rightarrow +\infty} \int \zeta |u_n|^2 dx = 0,$$

which contradicts (3.5).

In the second case (low frequency), we can assume (after extracting a subsequence) that $\tau_n \rightarrow \tau$ and (3.7) shows that the sequence $(u_n|_{B_{3/2}})$ is bounded in $H^1(B_{3/2})$. Hence, (after taking a subsequence), we can assume that it converges weakly in $H^1(B_{3/2})$ (and hence strongly in $L^2(B_{3/2})$). Due to the convergence of the family of metrics, we get

$$-\Delta_{g_{\sigma_n}} u_n = -\Delta_0 u_n + o(1)_{H^{-1}}, \quad (\Delta_0 = \sum_{i=1}^k \partial_j^2),$$

and according to (3.5) this implies that the limit u satisfies

$$(-\Delta_0 - \tau)u = 0 \text{ in } \mathcal{D}'(B_{3/2}), \quad u|_{1 < |z| < 3/2} = 0.$$

Uniqueness for solutions of second order elliptic operators implies that $u = 0$ which is contradictory with the strong convergence of (u_n) in $L^2(B_{3/2})$ and (3.5).

Finally it remains to study the last case (hyperbolic). Taking a subsequence, we can assume $\tau_n \rightarrow +\infty$. Moreover dividing both members of (3.4) by τ_n we see that u_n is solution of an equation of type $(P(x, \tau_n^{-\frac{1}{2}} D_x) - 1)u_n = \tau_n^{-1} f_n \rightarrow 0$ in $L^2(B(0, 2))$. The sequence $(u_n|_{|x| < 3/2})$ has a semi-classical measure ν with scale

$$\tilde{h}_n = \tau_n^{-1/2},$$

(see the end of Section 2 for a few facts about these measures). Notice that this new semi-classical parameter \tilde{h}_n has no relationship with the parameter h in Theorem 1.1. First of all multiplying both sides of (3.7) by $\tilde{h}_n^2 = \tau_n^{-1}$ and using the fact that $\|u_n\|_{L^2(B_2)}$ is uniformly bounded we deduce that there exists $C > 0$ such that

$$(3.8) \quad \tilde{h}_n \|\nabla_x u_n\|_{L^2(B_{3/2})} \leq C, \quad \forall n \in \mathbb{N}.$$

Using again (3.7) shows that the sequence $u_n|_{|x| < 3/2}$ is \tilde{h}_n -oscillatory (and hence the measure ν has total mass $1 = \lim_{n \rightarrow +\infty} \|u_n\|_{L^2(B_{3/2})}^2$). Now setting $D_n = \det((g_{\sigma_n})_{ij})$ we can write

$$(3.9) \quad \Delta_{g_{\sigma_n}} = \Delta_0 + \sum_{i,j=1}^k \partial_i \{ (g_{\sigma_n}^{ij} - \delta_{ij}) \partial_j \} + \frac{1}{2D_n} \sum_{i,j=1}^k g_{\sigma_n}^{ij} (\partial_i D_n) \partial_j.$$

The only point of importance below will be that

$$(3.10) \quad \lim_{n \rightarrow +\infty} \|g_{\sigma_n}^{ij} - \delta_{ij}\|_{W^{1,\infty}(B_2)} = 0, \quad \lim_{n \rightarrow +\infty} \|D_n - 1\|_{W^{1,\infty}(B_2)} = 0.$$

Proposition 3.4. — *The measure ν is supported in the set $\{(x, \zeta) : |\zeta| = 1\}$ and is invariant by the bicharacteristic flow associated to the metric g_0 :*

$$2\xi \cdot \nabla_x \nu = 0.$$

The contradiction now follows since by (3.5) we have $\|u_n\|_{L^2(1 < |z| < 2)} \rightarrow 0$ which implies that $\nu|_{1.1 < |x| < 1.9} = 0$ and by propagation that $\nu|_{|x| < 3/2} = 0$. It remains to prove Proposition 3.4.

Proof. — We have for a with compact support (in the x variable) in $B(0, 2)$,

$$(3.11) \quad \begin{aligned} & (a(x, \tilde{h}_n D_x)(\tilde{h}_n^2 \Delta_{g_{\sigma_n}} - 1)u_n, u_n)_{L^2} = (1) + (2) + (3), \\ & (1) = (a(x, \tilde{h}_n D_x)(\tilde{h}_n^2 \Delta_0 - 1)u_n, u_n)_{L^2}, \\ & (2) = \sum_{i,j} ((g_{\sigma_n}^{ij} - \delta_{ij}) \tilde{h}_n \partial_j u_n, \tilde{h}_n \partial_i a^*(x, \tilde{h}_n D_x)u_n)_{L^2}, \\ & (3) = \tilde{h}_n \sum_{ij} \left(\frac{1}{2D_n} (g_{\sigma_n}^{ij} \partial_i D_n \tilde{h}_n \partial_j u_n, a^*(x, \tilde{h}_n D_x)u_n)_{L^2} \right). \end{aligned}$$

On one hand, using the symbolic calculus, the term (1) tends to

$$\langle \nu, (|\zeta|^2 - 1)a(x, \zeta) \rangle.$$

Now using (3.8) and (3.10) we see easily that the terms (2) and (3) tend to zero when $n \rightarrow +\infty$. On the other hand, the l.h.side in (3.11) is equal to

$$\tilde{h}_n^2 (a(x, \tilde{h}_n D_x)u_n, u_n)_{L^2}.$$

and according to (3.5) tends to 0. We deduce

$$\forall a \in C_0^\infty(\mathbb{R}^{2k}), \langle \nu, (|\zeta|^2 - 1)a(x, \zeta) \rangle \Rightarrow \text{supp } (\nu) \subset \{(x, \zeta); |\zeta|^2 = 1\}.$$

To prove the second part in Proposition 3.4, we shall use the following lemma

Lemma 3.5. — *Let $a \in C_0^\infty(\mathbb{R}^{2k})$, and $b \in W^{1,\infty}(\mathbb{R}^{2k})$. Then*

$$(3.12) \quad \|[a(x, \tilde{h}_n D_x), b]\|_{\mathcal{L}(L^2)} \leq C \tilde{h}_n \|\nabla_x b\|_{L^\infty}.$$

Proof. — The kernel of the operator $[a(x, \tilde{h}_n D_x), b]$ is equal to (here $\phi \in C_0^\infty(\mathbb{R}^k)$ is equal to 1 on the x -projection of the support of a)

$$K(x, x') = \frac{1}{(2\pi \tilde{h}_n)^k} \int_{\zeta \in \mathbb{R}^k} e^{\frac{i}{\tilde{h}_n} \zeta \cdot (x - x')} a(x, \zeta) (b(x) - b(x')) \phi(x') d\zeta,$$

which is for $|x - x'| \leq \tilde{h}_n$ (since the support of a is compact) bounded by

$$(3.13) \quad C \tilde{h}_n^{-k} \|\nabla_x b\|_{L^\infty} |x - x'|,$$

while for $|x - x'| \geq \tilde{h}_n$ we can integrate by parts using the identity

$$\frac{\tilde{h}_n(x - x')}{i|x - x'|^2} \cdot \nabla_\zeta (e^{\frac{i}{\tilde{h}_n} \zeta \cdot (x - x')}) = e^{\frac{i}{\tilde{h}_n} \zeta \cdot (x - x')},$$

which gives

$$K(x, x') = \frac{1}{(2\pi\tilde{h}_n)^k} \int_{\zeta \in \mathbb{R}^k} e^{\frac{i}{\tilde{h}_n} \zeta \cdot (x-x')} \left(\tilde{h}_n \frac{(x-x') \cdot \nabla_\zeta}{i|x-x'|^2} \right)^N a(x, \zeta) (b(x) - b(x')) \phi(x') d\zeta,$$

and hence gives the bound for any $N \in \mathbb{N}$,

$$(3.14) \quad |K(x, x')| \leq \frac{C_N \tilde{h}_n^{N-k}}{|x-x'|^{N-1}} \|\nabla_x b\|_{L^\infty}.$$

It follows from (3.13) and (3.14) that

$$\int_{\mathbb{R}^k} |K(x, x')| dx + \int_{\mathbb{R}^k} |K(x, x')| dx' \leq C \tilde{h}_n \|\nabla_x b\|_{L^\infty}.$$

Then Lemma 3.5 follows from Schur's lemma. \square

Denoting by $[A, B]$ the commutator of the operators A and B let us set

$$C = \frac{i}{\tilde{h}_n} ([a(x, \tilde{h}_n D_x), (\tilde{h}_n^2 \Delta_{g_{\sigma_n}} - 1)] u_n, u_n)_{L^2}.$$

Then we can write using (3.4) and (3.9),

$$(3.15) \quad \begin{aligned} C &= \frac{i}{\tilde{h}_n} ([a(x, \tilde{h}_n D_x), \tilde{h}_n^2 f_n], u_n)_{L^2} = (1) + (2) + (3) \\ (1) &= \frac{i}{\tilde{h}_n} ([a(x, \tilde{h}_n D_x), (\tilde{h}_n^2 \Delta_0 - 1)] u_n, u_n)_{L^2}, \\ (2) &= \frac{i}{\tilde{h}_n} \sum_{j,l=1}^k ([a(x, \tilde{h}_n D_x), \tilde{h}_n \partial_j (g_{\sigma_n}^{jl} - \delta_{jl}) \tilde{h}_n \partial_l]) u_n, u_n)_{L^2}, \\ (3) &= \frac{i}{\tilde{h}_n} \sum_{j,l=1}^k \tilde{h}_n ([a(x, \tilde{h}_n D_x), \frac{1}{2D_n} g_{\sigma_n}^{jl} (\partial_j D_n) \tilde{h}_n \partial_l]) u_n, u_n)_{L^2}. \end{aligned}$$

By symbolic calculus, the term (1) is modulo an $\mathcal{O}(\tilde{h}_n)$ term equal to

$$(\text{Op}(\{a(x, \zeta), |\zeta|^2\}) u_n, u_n)_{L^2},$$

where $\{, \}$ denotes the Poisson bracket, and hence tends to

$$\langle \nu, \{a(x, \zeta), |\zeta|^2\} \rangle = \langle 2\zeta \cdot \nabla_x \nu, a \rangle.$$

Let us look to (2). Each term in the sum can be bounded by

$$\begin{aligned} &\frac{1}{\tilde{h}_n} |([a(x, \tilde{h}_n D_x), \tilde{h}_n \partial_j] (g_{\sigma_n}^{jl} - \delta_{jl}) \tilde{h}_n \partial_l u_n, u_n)_{L^2}| \\ &\quad + \frac{1}{\tilde{h}_n} |([a(x, \tilde{h}_n D_x), g_{\sigma_n}^{jl} - \delta_{jl}] \tilde{h}_n \partial_l u_n, \tilde{h}_n \partial_j u_n)| \\ &\quad + \frac{1}{\tilde{h}_n} |((g_{\sigma_n}^{jl} - \delta_{jl}) [a(x, \tilde{h}_n D_x), \tilde{h}_n \partial_l] u_n, \tilde{h}_n \partial_j u_n)|. \end{aligned}$$

By the semiclassical symbolic calculus and Lemma 3.5 the norms in $\mathcal{L}(L^2)$ of the operators $[a(x, \tilde{h}_n D_x), \tilde{h}_n \partial_j]$ and $[a(x, \tilde{h}_n D_x), g_{\sigma_n}^{jl} - \delta_{jl}]$ are bounded respectively by $C\tilde{h}_n$ and $C\tilde{h}_n \|\nabla_x g_{\sigma_n}^{jl}\|_{L^\infty}$ where C is independent of n . Therefore using (3.10) and (3.8) we deduce that (2) tends to zero when n goes to $+\infty$.

Unfolding the commutator and using (3.5), (3.8) we see that the third term in (3.15) is a finite sum of terms which are bounded by $C\|\partial_j D_n\|_{L^\infty}$. We deduce from (3.10) that (3) tends to zero when n goes to $+\infty$.

Now, opening the commutator we see that the r.h.s. in the first equation in (3.15) is equal to

$$\frac{i}{\tilde{h}_n} (a(x, \tilde{h}_n D_x) \tilde{h}_n^2 f_n, u_n)_{L^2} - \frac{i}{\tilde{h}_n} (\tilde{h}_n^2 f_n, a^*(x, \tilde{h}_n D_x) u_n)_{L^2}.$$

These terms are bounded by $C\tilde{h}_n\|f_n\|_{L^2(B_2)}\|u_n\|_{L^2(B_2)}$ and tend to zero when n goes to $+\infty$ since, according to (3.5), $\|u_n\|_{L^2(B_2)}$ is uniformly bounded and $\|f_n\|_{L^2(B_2)} = o(\tau_n^{\frac{1}{2}}) = o(\tilde{h}_n^{-1})$. This ends the proof of Lemma 3.4, and hence of Proposition 3.2 \square

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